

Deformations of hyperbolic convex polyhedra and 3-cone-manifolds

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Abstract

The Stoker problem, first formulated in [12], consists in understanding to what extent a convex polyhedron is determined by its dihedral angles. By means of the double construction, this problem is intimately related to rigidity issues for 3-dimensional cone-manifolds. In [9], two such rigidity results were proven, implying that the infinitesimal version of the Stoker conjecture is true in the hyperbolic and Euclidean cases. In this second article, we prove that local rigidity holds and obtain that the space of convex hyperbolic polyhedra with given combinatorial type is locally parameterized by the set of dihedral angles, together with a similar statement for hyperbolic 3-cone-manifolds.

Résumé

Le problème de Stoker, formulé pour la première fois dans [12], consiste à comprendre dans quelle mesure un polyèdre convexe est déterminé par ses angles dièdres. Via la construction qui à un polyèdre associe son double, ce problème est intimement lié à des questions de rigidité pour les cônes-variétés de dimension 3. Dans [9], deux tels résultats de rigidité ont été prouvés, impliquant que la version infinitésimale de la conjecture de Stoker est vraie dans les cas euclidien et hyperbolique. Dans ce deuxième article, nous passons de l'infinitésimal au local, et montrons que l'espace des polyèdres hyperboliques convexes de combinatoire fixée est localement paramétré par la donnée des angles dièdres, ainsi qu'un résultat similaire pour les cônes-variétés hyperboliques.

1 Introduction

In his 1968 article [12], Stoker asks the following question: if \mathcal{P} is a convex polyhedron, then is it true that the internal angles of its faces are determined by the dihedral angles of its edges? This conjecture, originally intended for Euclidean polyhedra, has been readily extended to convex polyhedra in the 3-sphere or the 3-dimensional hyperbolic space. In both latter cases, the question becomes whether a spherical or hyperbolic convex polyhedron is determined by its combinatorial type and its dihedral angles. In a first article [9], an infinitesimal version of the Stoker problem was proven in the Euclidean and hyperbolic case. It states that there is no nontrivial deformation of a convex hyperbolic polyhedron for which the infinitesimal variation of all dihedral angles vanishes; for a convex polyhedron in Euclidean space, such first-order deformations exist but preserve the internal angles of the faces.

This theorem is actually formulated in the more general setting of cone-manifolds. A hyperbolic or Euclidean 3-cone-manifold is a constant curvature stratified space, which can be locally described as a gluing of (hyperbolic or Euclidean) tetrahedra. The metric is smooth everywhere except on the singular locus, consisting of glued edges and vertices. Near a singular edge, the metric looks asymptotically like the product of an interval with a 2-dimensional cone, allowing to define the cone angle of this edge (a more precise definition is given in section 2.1). The link with polyhedra is straightforward : given a polyhedron \mathcal{P} , one can construct its double, which has a natural 3-cone-manifold structure. If \mathcal{P} is convex, then the cone angles of its double are smaller than 2π ; this restriction will always be present in this article.

Theorem 1 (The Infinitesimal Stoker Conjecture for Cone-manifolds, [9]).

Let \bar{M} be a closed, orientable three-dimensional cone-manifold with all cone angles smaller than 2π . If \bar{M} is hyperbolic, then \bar{M} is infinitesimally rigid relative to its cone angles, i.e. every angle-preserving infinitesimal deformation is trivial. If \bar{M} is Euclidean, then every angle-preserving deformation also preserves the spherical links of the codimension 3 singular points of \bar{M} .

In particular, convex hyperbolic polyhedra are infinitesimally rigid relative to their dihedral angles, while every dihedral angle preserving infinitesimal deformation of a convex Euclidean polyhedron also preserves the internal angles of the faces.

The goal of this article is to show a local rigidity result in the hyperbolic case, for closed cone-manifolds (with cone angles smaller than 2π) and convex polyhedra. The fact that infinitesimal rigidity implies local rigidity in this setting was already proven by Hodgson and Kerckhoff in [5] and by Weiss in [15], respectively in the case where the singular locus is a link and in the case where the cone angles are smaller than π (in both papers, the authors also prove the infinitesimal rigidity). Actually, the technique in section 6 of Weiss's article is valid as long as the singular locus is a trivalent graph. The main difficulty encountered in this paper is that for a non-trivalent singular locus, "splitting" of vertices may occur, consider for instance the double of the pictured polyhedron. But note that the singular locus changes.

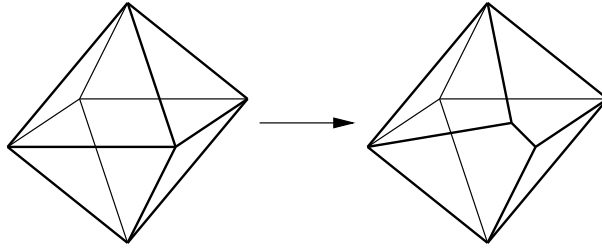


Figure 1: Splitting of a vertex

We will show in this article how to circumvent this difficulty and prove the following parameterization theorem (compare with Theorem 4.7 of [5] and Corollary 1.3 of [15]):

Theorem 2. *Let \bar{M} be a closed, orientable, hyperbolic 3-cone-manifold with singular locus Σ and whose cone angles are smaller than 2π . Then the space of hyperbolic 3-cone-manifold structures with singular locus Σ near \bar{M} is locally parameterized by the tuple of cone angles.*

The application to convex hyperbolic polyhedra is then elementary, and a strong version of the local hyperbolic case of the Stoker problem holds:

Theorem 3. *The space of convex hyperbolic polyhedra with given combinatorial type is locally parameterized by the tuple of dihedral angles.*

This result leaves some questions open. Firstly, it does not imply global rigidity, namely, the congruence of two convex hyperbolic polyhedra having same combinatorial type and dihedral angles. It is well known when the dihedral angles are non-obtuse, i.e. smaller than $\pi/2$: this is the famous Andreev's Theorem [1], see also [10] for a corrected and allegedly more readable proof. But it remains unsolved otherwise, not to mention the case where some dihedral angles are bigger than π . The same situation happens for hyperbolic 3-cone-manifolds. Global rigidity is known to hold only when all cone angles are smaller than π , see [8] and [16], and except in special cases (cf. [6]), almost nothing is known for local or infinitesimal rigidity when some cone angles are bigger than 2π .

Secondly, an analogous local result for Euclidean convex polyhedra or cone-manifolds would be very interesting. But the technique used here can certainly not be applied directly, since in the Euclidean

case angle-preserving infinitesimal deformations do exist; actually, even what the statement should be is not quite obvious. On the other hand, the Stoker problem is known to be false in the spherical case, see [11].

The outline of this article is as follows. In section 2, we begin by giving a precise definition of a hyperbolic 3-cone-manifold, then we review classic material about deformations of hyperbolic structure and see what it means for cone-manifolds. In particular, we reformulate the infinitesimal rigidity theorem of [9] in this formalism. Roughly speaking, a hyperbolic structure on a 3-manifold M is locally determined by the conjugacy class of its holonomy representation, i.e. a homomorphism $\rho : \pi_1(M) \rightarrow PSL(2, \mathbb{C})$. The space of infinitesimal (i.e. first-order) deformations of this structure is thus identified with the group cohomology space $H^1(\pi_1(M); Ad \circ \rho)$, which in turn can be identified with the cohomology group $H^1(M; \mathcal{E})$ for 1-forms with value in a geometric vector bundle \mathcal{E} . The infinitesimal rigidity in this setting corresponds to vanishing results for these cohomology spaces.

The fact that M is the regular part of a cone-manifold with given singular locus has important consequences on its holonomy representation ρ , which are best seen on the induced representation $i^*\rho$ on the boundary of a tubular neighborhood of the singular locus. In section 3, we study this kind of surface group representations and their deformations. As a result, we obtain on the space of representations a system of local coordinates near $i^*\rho$. In section 4, we show how to lift these local coordinates to the space of hyperbolic structures near M ; this yields the wanted parameterization on the subset of cone-manifolds with given singular locus. The application to the Stoker problem is then an easy corollary.

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2 Deformation theory of hyperbolic cone-manifolds and infinitesimal rigidity

2.1 Hyperbolic 3-cone-manifolds

There exist several ways to define cone-manifolds, depending on which aspect the author wants to emphasize. In this article, as we do not need a whole general theory, we will use somewhat simplified definitions, following Thurston [14]. The interested reader can refer to [9] for a more detailed approach.

A spherical cone-surface \bar{S} is a 2-dimensional singular Riemannian manifold, such that each point admits a neighborhood in which the expression of the metric g in polar coordinates is

$$g = dr^2 + \sin(r)^2 d\theta^2, \quad r \in [0, \epsilon), \quad \theta \in \mathbb{R}/\alpha\mathbb{Z}. \quad (1)$$

If α is equal to 2π , then this neighborhood is isometric to a disc in the two-sphere \mathbb{S}^2 : such a point is called regular. Otherwise, the point is singular and the quantity α is called the cone angle of this singular point. The set of regular points, called the regular part of \bar{S} , is easily seen to be an open dense subset of \bar{S} ; the singular points are isolated and form the singular locus of \bar{S} . In this article, we will mainly focus on orientable cone-manifolds with cone angles smaller than 2π ; the Gauss-Bonnet formula implies that such a spherical cone-surface is topologically the 2-sphere \mathbb{S}^2 . When it has only two singular points, the metric g can be expressed globally as

$$g = dr^2 + \sin(r)^2 d\theta^2, \quad r \in [0, \pi], \quad \theta \in \mathbb{R}/\alpha\mathbb{Z}; \quad (2)$$

this type of spherical cone-surfaces is relevant thereafter.

Likewise, a hyperbolic 3-cone-manifold is a 3-dimensional singular Riemannian manifold, such that each point admits a neighborhood in which the expression of the metric g in spherical coordinates is

$$g = dr^2 + \sinh(r)^2 g_{\bar{L}}, \quad r \in [0, \epsilon) \quad (3)$$

where $g_{\bar{L}}$ is the singular metric of a spherical cone-surface \bar{L} , called the link of the point; for the sake of simplicity we will require that \bar{L} is closed, connected and orientable. As in the surface case, if \bar{L} is the standard 2-sphere, then this neighborhood is isometric to a ball in the hyperbolic space \mathbb{H}^3 , and the point is called regular; otherwise it is called singular. Now if \bar{L} is a topological 2-sphere with only two cone points as in (2), the metric g can also be expressed locally in cylindrical coordinates as

$$g = d\rho^2 + \sinh(\rho)^2 d\theta^2 + \cosh(\rho)^2 dz^2, \quad \rho \in [0, \epsilon), \quad \theta \in \mathbb{R}/\alpha\mathbb{Z}, \quad z \in (-\epsilon, \epsilon) \quad (4)$$

In this coordinate system, the points of the set $\{\rho = 0\}$ are singular and share the same local expression of the metric. Their union is called a singular edge and the quantity α is called the cone angle (or dihedral angle) of this singular edge. The union of the singular edges forms an open, dense, 1-dimensional subset of the singular locus. The remaining singular points are isolated and are called singular vertices; topologically, the singular locus is a graph, geodesically embedded in the cone-manifold.

As mentioned in the introduction, the following construction is central to this article. Given a hyperbolic polyhedron \mathcal{P} (without “removable edges”, i.e. edges with dihedral angles equal to π), we can construct its double by gluing together \mathcal{P} and its mirror image along matching faces. This double is precisely a hyperbolic 3-cone-manifold; its singular locus corresponds to the edges and vertices of \mathcal{P} , and its cone angles are exactly twice the dihedral angles of \mathcal{P} . It is this construction that allows to translate statements relative to cone-manifolds into statements relative to polyhedra. Note that if \mathcal{P} is convex, then its dihedral angles are smaller than π , thus the cone angles of its double are smaller than 2π .

2.2 Holonomy representation and developing map

Let \bar{M} be a connected, orientable hyperbolic 3-cone-manifold and denote by M (resp. Σ) its regular part (resp. singular locus). Then M is an incomplete hyperbolic 3-manifold, whose metric completion is exactly \bar{M} , and we can apply to M the classic machinery of geometric structures (see [4] for a thorough exposition of the subject, and also [5] for the case of hyperbolic 3-cone-manifolds).

Let \tilde{M} denote the universal cover of M and $\pi : \tilde{M} \rightarrow M$ the associated projection; the hyperbolic metric g on M lifts to a hyperbolic metric $\tilde{g} = \pi^*g$ on \tilde{M} , and π becomes a local isometry. Let us choose a base-point \tilde{x} in $\pi^{-1}(x)$. Then the action of $\pi_1(M, x)$ on \tilde{M} via deck transformations is well-defined; it is transitive on the fibers. The hyperbolic metric \tilde{g} on \tilde{M} allows to define by analytical continuation the developing map $dev : \tilde{M} \rightarrow \mathbb{H}^3$, which is a local isometry, well-defined up to an isometry of \mathbb{H}^3 (acting by left composition). In particular $\tilde{g} = dev^*g_{\mathbb{H}^3}$, and the metric on M is completely determined by the developing map and the projection π . The developing map clearly features an equivariant property: there exists an application $hol : \pi_1(M, x) \rightarrow \text{Isom}^+(\mathbb{H}^3)$, called the holonomy representation, such that for all $p \in \tilde{M}$ and $\gamma \in \pi_1(M, x)$,

$$dev(\gamma.p) = hol(\gamma).dev(p).$$

The holonomy representation is well-defined up to conjugation by an isometry of \mathbb{H}^3 . Note that contrarily to the complete case, it has no reason to be faithful nor discrete.

Let us denote by $R(\pi_1(M, x), \text{Isom}^+(\mathbb{H}^3))$ the representation space, i.e. the set of all group homomorphisms from $\pi_1(M, x)$ to $\text{Isom}^+(\mathbb{H}^3)$, endowed with the compact-open (or pointwise convergence)

topology. Denote also by $X(\pi_1(M, x), \text{Isom}^+(\mathbb{H}^3))$ the quotient of this representation space by the action of $\text{Isom}^+(\mathbb{H}^3)$ by conjugation. The holonomy representation being determined, up to conjugation, by the hyperbolic metric g on M , we get a map

$$\{\text{hyperbolic metrics on } M\} \rightarrow X(\pi_1(M, x), \text{Isom}^+(\mathbb{H}^3)). \quad (5)$$

Now if M is the regular part of an oriented cone-manifold, its holonomy representation has some additional properties. If p is a singular point of \bar{M} , denote by L_p the regular part of its spherical link and by N_p the regular part of small enough neighborhood of p ; then L_p is a deformation retract of N_p . The inclusion map $i : N_p \hookrightarrow M$ induces a representation $i^* \text{hol} : \pi_1(N_p) \simeq \pi_1(L_p) \rightarrow \text{Isom}(\mathbb{H}^3)$. The image of $i^* \text{hol}$ is then contained in a maximal compact subgroup K of $\text{Isom}^+(\mathbb{H}^3)$: the induced representation is indeed (via the identification $K \simeq SO(3)$) the holonomy representation of the spherical structure on L_p . This observation is of course most relevant when p is a singular vertex of Σ . When applied to a point in a singular edge e , it shows that if $\gamma \in \pi_1(M, x)$ is freely homotopic to a meridian around e , then $\text{hol}(\gamma)$ is an elliptic isometry, whose rotation angle is equal modulo 2π to the cone angle of this singular edge.

2.3 Local deformation of a hyperbolic structure

Let g_1 and g_2 be two incomplete hyperbolic metrics on an orientable 3-manifold M , whose metric completions \bar{M}_1 and \bar{M}_2 are cone-manifolds. We will say that these two hyperbolic cone-manifolds are equivalent if there exists a diffeomorphism ϕ of M , isotopic to the identity, such that $g_1 = \phi^* g_2$. A cone-manifold structure is then an equivalence class for this relation. More generally, we can define the same equivalence relation for any hyperbolic metrics on M ; we will denote the quotient space (or hyperbolic structure space) by \mathcal{M} . Since two equivalent metrics induce the same (up to conjugation) holonomy representation, we can quotient (5) as a map

$$\mathcal{M} \rightarrow X(\pi_1(M, x), \text{Isom}^+(\mathbb{H}^3)). \quad (6)$$

There is another equivalence relation we need to introduce, namely the one induced by thickening. For simplicity, we will formulate it when M is diffeomorphic to the interior of a compact manifold with boundary, which is always the case if M is the regular part of a cone-manifold. Then M is diffeomorphic to $M \amalg \partial M \times [0, \epsilon)$. Given a hyperbolic metric g on M , this diffeomorphism pulls it to a metric on $M \amalg \partial M \times [0, \epsilon)$; let us denote its restriction to M by g' . The metric g is called a thickening of g' , and both give rise to the same (up to conjugation) holonomy representation. Let us denote by \sim the induced equivalence relation on \mathcal{M} ; we obtain a natural map

$$[\text{hol}] : \mathcal{M}_{/\sim} \rightarrow X(\pi_1(M, x), \text{Isom}^+(\mathbb{H}^3)). \quad (7)$$

We would like this map to be a homeomorphism, but actually this is not the case unless we restrict it to irreducible representations. More precisely, we have the following result, which is true in a much more general framework:

Theorem 4 (Deformation Theorem, see [4] section 3).

Let $\mathcal{R} = [\text{hol}]^{-1}(X^{\text{irr}}(\pi_1(M, x), \text{Isom}^+(\mathbb{H}^3))) \subset \mathcal{M}_{/\sim}$. Then the map

$$[\text{hol}] : \mathcal{R} \rightarrow X^{\text{irr}}(\pi_1(M), \text{Isom}^+(\mathbb{H}^3))$$

is a local homeomorphism.

This important result shows that in order to study the local deformation of a hyperbolic structure, it is sufficient to understand the local structure of the irreducible part of the quotient representation space $X(\pi_1(M, x), \text{Isom}^+(\mathbb{H}^3))$.

This space can actually be studied in a quite general setting. For any discrete group Γ and any Lie group G , we can consider the set $R(\Gamma, G)$ of all homomorphisms from Γ to G ; such an homomorphism is called a representation of Γ . If Γ admits a finite presentation $\langle s_1, \dots, s_n \mid f_1(s_1, \dots, s_n), \dots, f_p(s_1, \dots, s_n) \rangle$ then its representation space $R(\Gamma, G)$ can be identified with the subset

$$\{(x_1, \dots, x_n) \in G^n ; f_i(x_1, \dots, x_n) = e \ \forall i = 1 \dots p\},$$

which is an algebraic variety as soon as G is an algebraic group. This identification allows to define the usual topology as well as the Zariski one on $R(\Gamma, G)$, the former being the pointwise convergence one. The group G acts on $R(\Gamma, G)$ by conjugation and we denote the quotient space by $X(\Gamma, G)$. Note that this quotient is not well-behaved in general: it is usually non-Hausdorff, having non-closed points. In the literature, one often encounter another space also denoted by $X(\Gamma, G)$, called the character variety, which is the algebraic-geometric quotient and whose elements correspond to the closure of conjugation classes in $R(\Gamma, G)$; it will not be used here.

Let $\rho : \Gamma \rightarrow G$ be a representation. A first-order (or infinitesimal) deformation of ρ is then a function $\dot{\rho} : \Gamma \rightarrow \mathfrak{g}$, where \mathfrak{g} is the Lie algebra of G , satisfying the cocycle condition

$$\dot{\rho}(\gamma_1 \gamma_2) = \dot{\rho}(\gamma_1) + \text{Ad}(\rho(\gamma_1))(\dot{\rho}(\gamma_2)) \quad \forall \gamma_1, \gamma_2 \in \Gamma \quad (8)$$

We denote by $Z^1(\Gamma, \text{Ad} \circ \rho)$ the space of all maps from Γ to \mathfrak{g} satisfying this cocycle condition; it is canonically identified with the Zariski tangent space of $R(\Gamma, G)$ at ρ .

A deformation of the representation ρ is called trivial if it corresponds to a conjugation, namely if it is of the form $\dot{\rho}(\gamma) = \frac{d}{dt}|_{t=0} g_t \rho(\gamma) g_t^{-1}$. This is equivalent to satisfying the coboundary condition: there exists $v \in \mathfrak{g}$ such that for all $\gamma \in \Gamma$,

$$\dot{\rho}(\gamma) = v - \text{Ad}(\rho(\gamma))(v).$$

We denote by $B^1(\Gamma, \text{Ad} \circ \rho)$ the space of all maps from Γ to \mathfrak{g} satisfying this coboundary condition. The first group cohomology space $H^1(\Gamma, \text{Ad} \circ \rho)$ is then defined as the quotient $Z^1(\Gamma, \text{Ad} \circ \rho) / B^1(\Gamma, \text{Ad} \circ \rho)$. It is canonically identified with the Zariski tangent space of $X(\Gamma, G)$ at the equivalence class of ρ .

If Γ is the fundamental group of a connected manifold M , we can relate this construction to more usual cohomology spaces. More precisely, we can define on M a vector bundle \mathcal{E} associated to ρ as follows. On the universal cover \tilde{M} of M we can consider the trivial bundle $\tilde{\mathcal{E}} = \tilde{M} \times \mathfrak{g}$; it has a trivial flat connexion D . Recall that $\pi_1(M, x)$ acts by deck transformations on \tilde{M} , transitively on the fibers. Then we can define \mathcal{E} as the quotient $(\tilde{M} \times \mathfrak{g}) / \sim$, where $(p, v) \sim (\gamma.p, \text{Ad} \circ \rho(\gamma)(v))$. The connexion on $\tilde{\mathcal{E}}$ descends to a flat connexion, still denoted D , on \mathcal{E} . The reason for introducing \mathcal{E} is because there exists a classic isomorphism

$$H^1(M; \mathcal{E}) \xrightarrow{\sim} H^1(\pi_1(M, x); \text{Ad} \circ \rho)$$

between \mathcal{E} -valued form cohomology and group cohomology; this isomorphism is given by integration of a closed \mathcal{E} -valued 1-form along loops in M . This allows to make use of geometric analysis techniques in the study of local and infinitesimal deformations.

For an orientable hyperbolic 3-manifold M , \mathcal{E} is a $\mathfrak{sl}(2, \mathbb{C})$ -bundle, and can be interpreted as the bundle of infinitesimal local isometries, or local Killing vector fields. The fiber over a point $x \in M$ corresponds to the vector space of germs at x of Killing vector fields, and flat sections of \mathcal{E} over an open set U corresponds to Killing vector fields on U . The vector bundle \mathcal{E} admits a geometric decomposition $\mathcal{P} \oplus \mathcal{K}$, where at a point x the fiber \mathcal{P}_x corresponds to “infinitesimal pure translations” through x (i.e. derivatives of, or Killing fields integrating as, hyperbolic isometries whose axis goes through x) and the fiber \mathcal{K}_x to infinitesimal rotations (i.e. elliptic isometries) fixing x ; the sub-bundle \mathcal{P} is naturally identified with the tangent bundle TM . Note that the flat connexion on \mathcal{E} does not preserve this decomposition.

An element of $H^1(M; \mathcal{E})$ will be called an infinitesimal deformation of the hyperbolic structure on M (or of M for short). Indeed, $H^1(M; \mathcal{E})$ is identified to the Zariski tangent space of $X(\pi_1(M), \text{Isom}^+(\mathbb{H}^3))$ at $[\rho]$. If $[\rho]$ is a smooth point of this quotient representation space, then this is the usual tangent space, and if ρ is irreducible then by Theorem 4 it corresponds to the tangent space to the set of hyperbolic structure on M . So elements of $H^1(M; \mathcal{E})$ are in bijection with first-order deformations of the hyperbolic structure on M .

2.4 Infinitesimal rigidity

We recall the following result from [9]:

Theorem 5. *Let \bar{M} be a closed, orientable, hyperbolic 3-cone-manifold with all cone angles smaller than 2π . Then \bar{M} is infinitesimally rigid relative to its cone angles, i.e. every angle-preserving infinitesimal deformation is trivial.*

Although it is not explicitly stated, this theorem only deals with infinitesimal deformations that preserve the singular locus, i.e. no splitting of vertex occurs. If it were the case, new singular edges would appear; a more general rigidity result would have to take their angles into account.

The above theorem was actually proven in the language of curvature-preserving infinitesimal deformations of the metric tensor:

Theorem 6. *Let \bar{M} be a closed, orientable, hyperbolic 3-cone-manifold with all cone angles smaller than 2π , and let $h \in \mathcal{S}^2 M$ be a curvature-preserving infinitesimal deformation of the hyperbolic cone-metric g . If h and ∇h are in L^2 , then h is trivial, i.e. there exists a vector field X such that h is equal to the Lie derivative $L_X g$.*

The relation between curvature-preserving infinitesimal deformations of the metric tensor and the formalism we have exposed earlier, i.e. deformations as closed \mathcal{E} -valued 1-forms, is very well described in [5]; for the sake of clarity we will outline this relation now. If ω is a closed \mathcal{E} -valued 1-form, we can lift it to a closed $\tilde{\mathcal{E}}$ -valued 1-form $\tilde{\omega}$ on \tilde{M} . Since \tilde{M} is simply connected, $\tilde{\omega}$ is exact and hence equal to $D\tilde{s}$ for some section \tilde{s} of $\tilde{\mathcal{E}}$. Similarly to \mathcal{E} , the fiber bundle $\tilde{\mathcal{E}}$ decomposes as $\tilde{\mathcal{P}} \oplus \tilde{\mathcal{K}}$, where $\tilde{\mathcal{P}} \simeq T\tilde{M}$. The $\tilde{\mathcal{P}}$ -part of \tilde{s} can thus be identified with a vector field \tilde{X} on \tilde{M} , which corresponds to the deformation of the developing map. In general \tilde{X} does not descend to a vector field on M , but it satisfies an equivariant property (it is “automorphic” in the language of [5]). As a consequence the infinitesimal deformation of the metric tensor $\tilde{h} = L_{\tilde{X}} \tilde{g}$ actually descends to a well-defined curvature-preserving deformation h on M . Alternatively, we can also consider the $\tilde{\mathcal{K}}$ -part of $\tilde{\omega}$; it identifies with a $T\tilde{M}$ -valued 1-form (but not closed in general). Using the isomorphism between $T\tilde{M}$ and $T^*\tilde{M}$ given by the metric \tilde{g} , it can be seen as a section of $T^{(0,2)}\tilde{M} = T^*\tilde{M} \otimes T^*\tilde{M}$. Its symmetric part \tilde{h} is then the infinitesimal deformation of the metric.

On the other hand, if h is a curvature-preserving infinitesimal deformation of g , then it is locally an infinitesimal isometry, i.e. it can be written locally as $L_X g$ for some local vector field X . The lift \tilde{h} of h to the universal cover \tilde{M} is then equal to $L_{\tilde{X}} \tilde{g}$ for some globally defined vector field \tilde{X} , which is automorphic. Now there exists a section \tilde{r} of $\tilde{\mathcal{K}}$ such that the differential $\tilde{\omega} = D\tilde{s}$ of the section $\tilde{s} = \tilde{X} + \tilde{r}$ descends to a (closed) \mathcal{E} -valued 1-form ω on M , which is the desired deformation. Such a section \tilde{r} can be found by “osculating” \tilde{X} .

For a hyperbolic cone-manifold \bar{M} with singular locus Σ , let \bar{U}_ϵ be a small enough tubular neighborhood of Σ . We will denote by M_ϵ its complement and by $U_\epsilon = \bar{U}_\epsilon \setminus \Sigma$ its regular part. Then M_ϵ is a manifold with boundary whose interior is diffeomorphic to M . We will denote its boundary by Σ_ϵ ; it is a deformation retract of U_ϵ . These notations will be used extensively in the remainder of

this article. The infinitesimal rigidity result of Theorem 6 has important consequences in terms of the cohomology groups of these spaces:

Proposition 7. *Under the assumptions of Theorem 6, the map $H^1(M, U_\epsilon; \mathcal{E}) \rightarrow H^1(M; \mathcal{E})$ is the zero map, and the map $H^1(M; \mathcal{E}) \rightarrow H^1(U_\epsilon; \mathcal{E})$ is injective with half-dimensional image.*

Proof. Let $\omega \in \Omega^1(M, \mathcal{E})$ be a closed \mathcal{E} -valued 1-form, equal to zero over U_ϵ . The corresponding infinitesimal deformation h of the metric tensor is also zero over U_ϵ and therefore h and ∇h are in L^2 . We can then apply Theorem 6, which shows that h is trivial; consequently ω is a coboundary. This proves the first part of the proposition, namely the vanishing of the map $H^1(M, U_\epsilon; \mathcal{E}) \rightarrow H^1(M; \mathcal{E})$.

The fact that this implies the second part has been already proven in [5] and [15]; here is how it goes. By Poincaré duality, we obtain that the dual map $H^2(M, U_\epsilon; \mathcal{E}) \rightarrow H^2(M; \mathcal{E})$ also vanishes. Now we can look at (part of) the long exact sequence of the pair (M, U_ϵ) :

$$H^1(M, U_\epsilon; \mathcal{E}) \xrightarrow{1} H^1(M; \mathcal{E}) \xrightarrow{2} H^1(U_\epsilon; \mathcal{E}) \xrightarrow{3} H^2(M, U_\epsilon; \mathcal{E}) \xrightarrow{4} H^2(M; \mathcal{E})$$

The maps (1) and (4) vanish, so (2) is injective and (3) is surjective. This implies that $\dim H^1(U_\epsilon; \mathcal{E}) = \dim H^1(M; \mathcal{E}) + \dim H^2(M, U_\epsilon; \mathcal{E})$, but the two dimensions at the right hand side are equal since by Poincaré duality $H^2(M, U_\epsilon; \mathcal{E}) \simeq (H^1(M; \mathcal{E}))^*$. \square

Let us denote by ρ the holonomy representation of M and by i the inclusion map $\Sigma_\epsilon \hookrightarrow M$. We have seen that $H^1(M; \mathcal{E})$ can be identified with the tangent space of $X(\pi_1(M), \text{Isom}^+(\mathbb{H}^3))$ at $[\rho]$. And since the inclusion of Σ_ϵ into U_ϵ is a deformation retract, $H^1(U_\epsilon; \mathcal{E})$ is equal to $H^1(\Sigma_\epsilon; \mathcal{E})$, which in turn can be identified (if Σ_ϵ is connected) with the tangent space of $X(\pi_1(\Sigma_\epsilon), \text{Isom}^+(\mathbb{H}^3))$ at the conjugacy class of the representation $i^*\rho$ induced by ρ on Σ_ϵ . With these identifications, the above map $H^1(M; \mathcal{E}) \rightarrow H^1(\Sigma_\epsilon; \mathcal{E})$ is exactly the tangent map at $[\rho]$ of the restriction map $i^* : X(\pi_1(M), \text{Isom}^+(\mathbb{H}^3)) \rightarrow X(\pi_1(\Sigma_\epsilon), \text{Isom}^+(\mathbb{H}^3))$.

Recall that according to Theorem 4, we want to study locally $X(\pi_1(M), \text{Isom}^+(\mathbb{H}^3))$; so we will begin by looking at $X(\pi_1(\Sigma_\epsilon), \text{Isom}^+(\mathbb{H}^3))$ and then use the above proposition to lift the results to $X(\pi_1(M), \text{Isom}^+(\mathbb{H}^3))$.

3 Representations of surface groups

It is a well-known fact that the group $\text{Isom}^+(\mathbb{H}^3)$ of direct isometries of the hyperbolic 3-space is isomorphic to the Lie group $PSL(2, \mathbb{C})$. We have found it easier to work instead with its universal cover $SL(2, \mathbb{C})$, and this is possible thanks to the following result of Culler:

Theorem 8 (Representations Lifting, [2]).

Let M be an orientable, not necessarily complete, hyperbolic 3-manifold and let $\rho : \pi_1(M) \rightarrow PSL(2, \mathbb{C})$ be its holonomy representation. Then ρ can be lifted to $SL(2, \mathbb{C})$, i.e. there exists $\tilde{\rho} : \pi_1(M) \rightarrow SL(2, \mathbb{C})$ such that $\rho = \pi \circ \tilde{\rho}$ where π is the projection $SL(2, \mathbb{C}) \rightarrow PSL(2, \mathbb{C})$.

In the following, we will only work with representations into $SL(2, \mathbb{C})$; since the projection map from $X(\Gamma, SL(2, \mathbb{C}))$ to $X(\Gamma, PSL(2, \mathbb{C}))$ is a local diffeomorphism, this will cause no difference.

This result is also true for a (possibly incomplete) spherical surface: its holonomy representation can be lifted from $SO(3, \mathbb{C})$ to $SU(2, \mathbb{C})$. So here as well we will only work with representations into $SU(2, \mathbb{C})$. Now the inclusion map $SU(2, \mathbb{C}) \hookrightarrow SL(2, \mathbb{C})$ induces an injective map $R(\Gamma, SU(2, \mathbb{C})) \rightarrow R(\Gamma, SL(2, \mathbb{C}))$. Two unitary representations are conjugated if and only if they are unitarily conjugated; in other words, there exists a well-defined injective map $X(\Gamma, SU(2, \mathbb{C})) \rightarrow X(\Gamma, SL(2, \mathbb{C}))$. Consequently we will often think of these two spaces as one inside the other. The following result shows that the representations that we will consider in this article are irreducible:

Theorem 9. *Let \bar{S} be a closed, orientable spherical cone-surface with at least three singular points and cone angles smaller than 2π , and let $\rho : \pi_1(S) \rightarrow SU(2, \mathbb{C})$ be (the lift of) its holonomy representation. Then ρ is irreducible. In particular, its conjugacy class $[\rho]$ is a smooth point of $X(\pi_1(S), SU(2, \mathbb{C}))$ and of $X(\pi_1(S), SL(2, \mathbb{C}))$*

Proof. Since all cone angles are smaller than 2π , by the Gauss-Bonnet formula the regular part S of \bar{S} is homeomorphic to a d -punctured sphere ($d \geq 3$); consequently its fundamental group is a free group on $d-1$ generators, generated by loops going around the singular points. Thus the representation space $R(\pi_1(S), G)$ can be identified with G^{d-1} , which is smooth, and it is easy to see that for $G = SU(2, \mathbb{C})$ or $SL(2, \mathbb{C})$, the action of G by conjugation on the subset of irreducible representations of $R(\pi_1(S), G)$ is proper. So it is enough to prove that ρ is irreducible.

We begin by noting that the regular part S of \bar{S} , although an incomplete Riemannian manifold, is geodesically connected, i.e. there exists a (shortest) geodesic between any pair of points of S . Indeed, \bar{S} is a complete compact length space, so there exists a shortest path γ in \bar{S} between any pair of points of S . But a path going through a cone point of angle smaller than 2π cannot be a shortest path, so γ is in fact contained in S and is thus a (Riemannian) geodesic.

Now choose a base point x in S such that the shortest geodesic from x to any cone point is unique, and consider the complement C of the cut locus of x . Via the embedding of C in \bar{S} and the developing map, we can construct a local isometry $f : C \rightarrow \mathbb{S}^2$. We claim that f is injective, and (the closure of) its image is thus a fundamental polygon for \bar{S} . Indeed, suppose y_1 and y_2 are two points of C such that $f(y_1) = f(y_2)$. For $i = 1$ or 2 , by definition of C the shortest geodesic γ_i from x to y_i is contained in C , so $f(\gamma_1)$ and $f(\gamma_2)$ are two geodesics in \mathbb{S}^2 with the same endpoints $f(x)$ and $f(y_1) = f(y_2)$. So either they are equal, and then $\gamma_1 = \gamma_2$ and so $y_1 = y_2$, or one of them, say $f(\gamma_1)$, goes through the antipode of $f(x)$, which is a conjugate point to $f(x)$ along $f(\gamma)$. This latter case implies that x has a conjugate point along γ_1 , which by definition cannot belong to C , and this is impossible since γ_1 is contained in C .

Finally, let p be a cone point of \bar{S} . There is a unique shortest geodesic from x to p , so p corresponds to a unique point, that we will denote by $f(p)$, in the boundary of $f(C) \subset \mathbb{S}^2$. Then the holonomy of the loop based at x and going around p is a rotation centered at $f(p)$, whose angle is equal to the cone angle at p . Now since $\pi_1(S, x)$ is generated by loops going around the cone points, it implies that the image of the holonomy representation ρ is generated by rotations centered at the $f(p_i)$, which are distinct points of \mathbb{S}^2 because f is injective. And since \bar{S} has at least three cone points, all these rotations cannot have the same axis, and hence ρ is irreducible. \square

If p is a singular vertex of a hyperbolic 3-cone-manifold, we can apply this result to the induced holonomy representation on the spherical link of p to obtain the following corollary:

Corollary 10. *Let \bar{M} be a hyperbolic 3-cone-manifold with cone angles smaller than 2π and such that its singular locus contains at least a singular vertex of valence greater than or equal to 3. Then its holonomy representation is irreducible.*

This statement is actually true for any finite volume hyperbolic 3-cone-manifolds (see[15], Lemma 6.35), but we will not need this fact here.

If $\gamma \in \pi_1(S)$ is a loop on a surface S , we can define a function $\text{tr}_\gamma : R(\pi_1(S), SL(2, \mathbb{C})) \rightarrow \mathbb{C}$ by $\text{tr}_\gamma(\varrho) = \text{tr}(\varrho(\gamma))$. Now if l is a closed curve on S , all the loops in $\pi_1(S)$ freely homotopic to l are conjugated to one another, and since the trace is invariant by conjugation we can still define the function tr_l as being equal to tr_γ for any $\gamma \in \pi_1(S)$ freely homotopic to l . The invariance of the trace by conjugation also means that $\text{tr}_\gamma = \text{tr}_l$ descends to a function (keeping the same notation) $\text{tr}_\gamma = \text{tr}_l : X(\pi_1(S), SL(2, \mathbb{C})) \rightarrow \mathbb{C}$. In the exact same way, we can define functions tr_γ and

$\text{tr}_l : X(\pi_1(S), SU(2, \mathbb{C})) \rightarrow \mathbb{R}$. These trace functions will give us local coordinates on these quotient representation spaces.

Proposition 11. *Let \bar{S}_d be an orientable, closed spherical cone-surface with d singular points ($d \geq 3$) of cone angles smaller than 2π , and let S_d be its regular part. Let $(\gamma_k)_{1 \leq k \leq d}$ be loops around the singular points, such that $\langle \gamma_1, \dots, \gamma_d \mid \gamma_1 \dots \gamma_d = 1 \rangle$ is a presentation of $\pi_1(S_d)$, and let $\rho : \pi_1(S_d) \rightarrow SU(2, \mathbb{C}) \subset SL(2, \mathbb{C})$ be (the lift of) the holonomy representation. Then there exist local real-valued functions $f_1, \dots, f_{2d-6}, g_1, \dots, g_{2d-6}$ on a neighborhood U of $[\rho]$ in $X(\pi_1(S_d), SL(2, \mathbb{C}))$ such that the family $(\Re \text{tr}_{\gamma_k}, \Im \text{tr}_{\gamma_k})_{1 \leq k \leq d} \cup (f_k, g_k)_{1 \leq k \leq 2d-6}$ is a local system of coordinates in U and that $X(\pi_1(S_d), SU(2, \mathbb{C})) \cap U = \{\Im \text{tr}_{\gamma_1} = \dots = \Im \text{tr}_{\gamma_d} = f_1 = \dots = f_{2d-6} = 0\}$.*

Proof. We remark first that by Gauss-Bonnet formula, S_d is a sphere with d holes, which justifies the existence of the given presentation of its fundamental group. We will begin by proving that the functions $(\text{tr}_{\gamma_k})_{1 \leq k \leq d}$, have linearly independent derivatives at $T_\rho R(\pi_1(S_d), SU(2, \mathbb{C}))$. Note that since γ_d is equal to $(\gamma_1 \dots \gamma_{d-1})^{-1}$ and the representations have values in $SU(2, \mathbb{C})$ or $SL(2, \mathbb{C})$, the function tr_{γ_d} is equal to $\text{tr}_{\gamma_1 \dots \gamma_{d-1}}$. As before, we will identify $R(\pi_1(S_d), SU(2, \mathbb{C}))$ to $SU(2, \mathbb{C})^{d-1}$ by associating to a representation ρ the $(d-1)$ -uple (m_1, \dots, m_{d-1}) where $m_k = \rho(\gamma_k)$ for $k = 1 \dots d$. Note also that since m_k is the holonomy around a singular point, it is different from the identity. With this identification, the functions tr_{γ_k} , $k = 1 \dots d-1$, are clearly linearly independent in a neighborhood of ρ .

Now suppose that there exists a dependence relation between the derivatives at $\rho \simeq (m_1, \dots, m_{d-1})$, namely $d \text{tr}_{\gamma_d} = \sum_{1 \leq k \leq d-1} \lambda_k d \text{tr}_{\gamma_k}$. This means that for all $(v_1, \dots, v_{d-1}) \in \mathfrak{su}(2, \mathbb{C})^{d-1}$, we have

$$\sum_{1 \leq k \leq d-1} \text{tr}(m_1 \dots m_{k-1} v_k m_k m_{k+1} \dots m_{d-1}) = \sum_{1 \leq k \leq d-1} \lambda_k \text{tr}(v_k m_k).$$

In particular, $\text{tr}(m_1 \dots m_{k-1} v m_k m_{k+1} \dots m_{d-1}) = \lambda_k \text{tr}(v m_k)$ for all k , $1 \leq k \leq d-1$, and all $v \in \mathfrak{su}(2, \mathbb{C})$. This implies that $m_k m_{k+1} \dots m_{d-1} m_1 \dots m_{k-1} - \lambda_k m_k$ is a scalar multiple of the identity, and so that m_k and $m_k m_{k+1} \dots m_{d-1} m_1 \dots m_{k-1}$ commute (hence also m_k and $m_{k+1} \dots m_{d-1} m_1 \dots m_{k-1}$). But this being true for all k , it implies that $m_{k+1} \dots m_{d-1} m_1 \dots m_{k-1} m_k = m_1 \dots m_{d-1} = m_d^{-1}$ for all k , and in particular m_k and m_d commute for all k , which is impossible since ρ is irreducible. The same argument can be used to show that the (complex-valued) functions tr_{γ_k} , $k = 1 \dots d$, have linearly independent (complex) derivatives at $T_\rho R(\pi_1(S_d), SL(2, \mathbb{C}))$.

Now we have seen (Theorem 9) that the conjugacy class $[\rho]$ is a smooth point of both representation spaces $X(\pi_1(S), SU(2, \mathbb{C}))$ and $X(\pi_1(S), SL(2, \mathbb{C}))$; in particular, the tangent space at $[\rho]$ is well-defined. The above trace functions are invariant by conjugation, so they descend to these two quotient spaces, and their (real- or complex-valued) derivatives at $[\rho]$ are still linearly independent. The fact that $[\rho]$ is a smooth point also implies that $X(\pi_1(S_d), SU(2, \mathbb{C}))$ is locally embedded in $X(\pi_1(S_d), SL(2, \mathbb{C}))$ as a half-dimensional manifold, of real dimension $3(d-1) - 3 = 3d - 6$. This means that we can complete the family $(\Re \text{tr}_{\gamma_k}, \Im \text{tr}_{\gamma_k})_{1 \leq k \leq d}$ into a real coordinate system with the required properties. \square

Let ρ be the holonomy representation of a closed, orientable hyperbolic 3-cone-manifold \bar{M} , with singular locus Σ and cone angles smaller than 2π . As in section 2.4, the boundary of a tubular neighborhood of Σ is a surface denoted by Σ_ϵ . It is not necessarily connected; its components correspond to those of Σ . Let Σ^c be a component of Σ which is not a circle, and denote by V , resp. E the set of its singular vertices, resp. edges. Let Σ_ϵ^c be the corresponding component of Σ_ϵ ; it is a surface of genus $g \geq 2$.

By a slight abuse of notation, ρ will also denote the induced representation on Σ_ϵ^c . Let $(\mu_e)_{e \in E}$ be a family of simple closed curves on Σ_ϵ^c going around the singular edges; they split Σ_ϵ^c into a family

$(S_v)_{v \in V}$ of $d(v)$ -punctured spheres, where $d(v)$ is the valence of the vertex v . Let us denote by i_v the inclusion $S_v \hookrightarrow \Sigma_\epsilon^c$. We have seen that the induced representation $i_v^* \rho$ on S_v is actually the holonomy representation of the spherical cone-surface structure on the link of v ; it satisfies the hypotheses of Proposition 11 where the γ_k can be chosen (up to free homotopy) among the μ_e . We will denote by f_k^v , $k = 1 \dots d(v)$, the corresponding local functions mentioned in Proposition 11; they can be pulled back to functions on $X(\pi_1(\Sigma_\epsilon^c), SL(2, \mathbb{C}))$ and on $R(\pi_1(\Sigma_\epsilon^c), SL(2, \mathbb{C}))$. We will keep the same notations for these functions and their lifts and/or pull-backs.

Theorem 12. *With the above notations, consider the following local function*

$$F^c : X(\pi_1(\Sigma_\epsilon^c), SL(2, \mathbb{C})) \rightarrow \mathbb{R}^{6g-6} = \mathbb{R}^{2|E|} \bigoplus_{v \in V} \mathbb{R}^{2d(v)-6}$$

$$[\varrho] \mapsto \left(\left(\Re \text{tr}_{\mu_e}(\varrho), \Im \text{tr}_{\mu_e}(\varrho) \right)_{e \in E}, \left(f_k^v(\varrho) \right)_{v \in V, 1 \leq k \leq 2d(v)-6} \right)$$

Then in a neighborhood of $[\rho]$, which is a smooth point of $X(\pi_1(\Sigma_\epsilon^c), SL(2, \mathbb{C}))$, the level sets of F^c are local half-dimensional submanifolds.

Proof. We will begin by proving that the derivatives $d\Im \text{tr}_{\mu_e}$, $d\Re \text{tr}_{\mu_e}$, and so on are \mathbb{R} -linearly independent on $T_\rho^* R(\pi_1(\Sigma_\epsilon^c), SL(2, \mathbb{C}))$.

As mentioned above, the family $(\mu_e)_{e \in E}$ splits Σ_ϵ^c into a family of punctured spheres $(S_v)_{v \in V}$, corresponding each to a vertex v of Σ_ϵ^c , and the induced representation on S_v satisfies the hypotheses of Proposition 11. We will denote by μ_k^v , $k = 1 \dots d(v)$, the simple closed curves corresponding to the boundary components of S_v ; for each k the image $i_v(\mu_k^v)$ is homotopic (up to orientation) to one curve of the family $(\mu_e)_{e \in E}$, but note that two boundary components of S_v may correspond to the same curve in Σ_ϵ^c .

The next step is to rebuild Σ_ϵ^c by gluing together the punctured spheres S_v obtained by cutting Σ_ϵ^c . Thus we construct a family $\Sigma_0, \Sigma_1, \dots, \Sigma_{|E|} = \Sigma_\epsilon^c$, where Σ_0 is one of the punctured spheres S_{v_0} , and where Σ_l is obtained from Σ_{l-1} by performing one of the following operations:

1. gluing a punctured sphere S_{v_l} to Σ_{l-1} along one of its boundary components,
2. gluing Σ_{l-1} along two of its boundary components.

The surface Σ_l is therefore obtained as a gluing of a certain sub-family of the family of punctured spheres $(S_v)_{v \in V}$; let us denote by $V_l \subset V$ the corresponding subset of indices, i.e. Σ_l is obtained from the family $(S_v)_{v \in V_l}$. For each l , we have inclusion maps $i_l : \Sigma_l \hookrightarrow \Sigma_\epsilon^c$, $i_{v,l} : S_v \hookrightarrow \Sigma_l$ if $v \in V_l$, and $i_{l-1,l} : \Sigma_{l-1} \hookrightarrow \Sigma_l$ if $l \geq 1$, satisfying the obvious compatibility relations.

For $v \in V_l$, the local functions $(f_k^v)_{1 \leq k \leq 2d(v)-6}$ on $X(\pi_1(S_v), SL(2, \mathbb{C}))$ can be pulled back to functions on $X(\pi_1(\Sigma_l), SL(2, \mathbb{C}))$ via $i_{v,l}^* : X(\pi_1(\Sigma_l), SL(2, \mathbb{C})) \rightarrow X(\pi_1(S_v), SL(2, \mathbb{C}))$, and the same is true for their lifts to the representation spaces. On Σ_l we also have a family $(\mu_k^l)_{1 \leq k \leq I_l}$ of curves, corresponding to the boundary components of Σ_l and to the curves along which the previous gluings have been done. Note that all the surfaces Σ_l , $0 \leq l < |E|$, are compact and have a non empty boundary; their fundamental groups are therefore free groups and hence $R(\pi_1(\Sigma_l), SL(2, \mathbb{C})) \simeq SL(2, \mathbb{C})^n$ is a smooth manifold. This means that the tangent space $T_{i_l^* \rho} R(\pi_1(\Sigma_l), SL(2, \mathbb{C}))$ is actually well-defined, except maybe for $l = |E|$; this case will be dealt with later.

We will now prove by induction on l that the family of functions $(\Re(\text{tr}_{\mu_k^l}), \Im(\text{tr}_{\mu_k^l}))_{1 \leq k \leq I_l} \cup (i_{v,l}^*(f_k^v))_{v \in V_l, 1 \leq k \leq 2d(v)-6}$ has linearly independent derivatives on $T_{i_l^* \rho} R(\pi_1(\Sigma_l), SL(2, \mathbb{C}))$. We already know that this is true for $l = 0$ (Proposition 11). For the inductive step, there are two cases to consider, depending on the type of gluing.

Case 1: Σ_l is obtained as a gluing of Σ_{l-1} and of a punctured sphere, denoted S_{v_l} , of the family $(S_v)_{v \in V}$; more precisely, a boundary component μ_1 of Σ_{l-1} is identified to a boundary component μ_2 of S_{v_l} . The simple closed curve μ_1 (resp. μ_2) belongs to the family $(\mu_k^{l-1})_{1 \leq k \leq l-1}$ (resp. $(\mu_k^{v_l})_{1 \leq k \leq d(v_l)}$); let k_l (resp. k'_l) be its corresponding index. We will denote by μ the resulting curve on Σ_l ; it belongs to the family $(\mu_k^l)_{1 \leq k \leq l}$. We have the following commutative diagram, where all maps are inclusions:

$$\begin{array}{ccccc} & & S_{v_l} & & \\ & \nearrow^{j'} & & \searrow^{i_{v_l,l}} & \\ \mathbb{S}^1 & & & & \Sigma_l \\ & \searrow_j & & \nearrow^{i_{l-1,l}} & \\ & & \Sigma_{l-1} & & \end{array}$$

The fundamental group $\pi_1(\Sigma_l)$ is obtained as an amalgamated product of $\pi_1(\Sigma_{l-1})$ and $\pi_1(S_{v_l})$, and $R(\pi_1(\Sigma_l), SL(2, \mathbb{C}))$ is equal to the fiber product of $R(\pi_1(\Sigma_{l-1}), SL(2, \mathbb{C}))$ and $R(\pi_1(S_{v_l}), SL(2, \mathbb{C}))$ over $R(\pi_1(\mathbb{S}^1), SL(2, \mathbb{C}))$:

$$R(\pi_1(\Sigma_l), SL(2, \mathbb{C})) \simeq \{(\sigma, \tau) \in R(\pi_1(\Sigma_{l-1}), SL(2, \mathbb{C})) \times R(\pi_1(S_{v_l}), SL(2, \mathbb{C})) ; j^*(\sigma) = j'^*(\tau)\}$$

where j and j' are the inclusion maps from the glued boundary component $\mu \simeq \mathbb{S}^1$ into Σ_{l-1} and S_{v_l} respectively.

So suppose there is a linear dependence relation between the derivatives:

$$\sum_{1 \leq k \leq l} (\lambda_k^1 d\Re(\text{tr } \mu_k^l) + \lambda_k^2 d\Im(\text{tr } \mu_k^l)) + \sum_{v \in V_l} \sum_{1 \leq k \leq 2d(v)-6} \lambda_{v,k} d(i_{v,l} * (f_k^v)) = 0.$$

Each of the functions above is the push-forward of a function on either $R(\pi_1(\Sigma_{l-1}), SL(2, \mathbb{C}))$ or $R(\pi_1(S_{v_l}), SL(2, \mathbb{C}))$, except for $\Re(\text{tr } \mu)$ and $\Im(\text{tr } \mu)$ which fall in both cases, thus we can rewrite the dependence relation as:

$$\begin{aligned} 0 = & a d\Re(\text{tr } \mu) + b d\Im(\text{tr } \mu) \\ & + i_{l-1,l} * \left(\sum_{\substack{1 \leq k \leq l-1 \\ k \neq k_l}} (\lambda_k^1 d\Re(\text{tr } \mu_k^{l-1}) + \lambda_k^2 d\Im(\text{tr } \mu_k^{l-1})) + \sum_{v \in V_{l-1}} \sum_{1 \leq k \leq 2d(v)-6} \lambda_{v,k} d(i_{v,l-1} * (f_k^v)) \right) \\ & + i_{v_l,l} * \left(\sum_{\substack{1 \leq k \leq d(v_l) \\ k \neq k'_l}} (\lambda_k^1 d\Re(\text{tr } \mu_k^{v_l}) + \lambda_k^2 d\Im(\text{tr } \mu_k^{v_l})) + \sum_{1 \leq k \leq 2d(v_l)-6} \lambda_{v_l,k} d f_k^{v_l} \right) \end{aligned} \quad (9)$$

and we also have

$$a d\Re(\text{tr } \mu) + b d\Im(\text{tr } \mu) = i_{l-1,l} * (a d\Re(\text{tr } \mu_{k_l}^{l-1}) + b d\Im(\text{tr } \mu_{k_l}^{l-1})) = i_{v_l,l} * (a d\Re(\text{tr } \mu_{k'_l}^{v_l}) + b d\Im(\text{tr } \mu_{k'_l}^{v_l})).$$

According to Proposition 11, the functions of the family $(\Re \text{tr } \mu_k^{v_l}, \Im \text{tr } \mu_k^{v_l})_{1 \leq k \leq d(v_l)} \cup (f_k^{v_l})_{1 \leq k \leq 2d(v_l)-6}$ have linearly independent derivatives on $T_{i_{v_l,l}^* \rho} R(\pi_1(S_{v_l}), SL(2, \mathbb{C}))$. Consequently, for each tangent vector $\dot{\sigma} \in T_{i_{l-1,l}^* \rho} R(\pi_1(\Sigma_{l-1}), SL(2, \mathbb{C}))$, we can find $\dot{\tau} \in T_{i_{v_l,l}^* \rho} R(\pi_1(S_{v_l}), SL(2, \mathbb{C}))$ such that $d f_k^{v_l}(\dot{\tau}) = 0$ for $1 \leq k \leq 2d(v_l) - 6$, $d\Im \text{tr } \mu_k^{v_l}(\dot{\tau}) = d\Re \text{tr } \mu_k^{v_l}(\dot{\tau}) = 0$ for $1 \leq k \leq d(v_l)$ (except for $k = k'_l$), and such that $d\Im \text{tr } \mu_{k'_l}^{v_l}(\dot{\tau}) = d\Im \text{tr } \mu_{k_l}^{l-1}(\dot{\sigma})$ and $d\Re \text{tr } \mu_{k'_l}^{v_l}(\dot{\tau}) = d\Re \text{tr } \mu_{k_l}^{l-1}(\dot{\sigma})$, i.e. $(\dot{\sigma}, \dot{\tau})$ belongs to $T_{i_l^* \rho} R(\pi_1(\Sigma_l), SL(2, \mathbb{C}))$. Applying this to (9) and pulling back, we find a dependence relation on $T_{i_{l-1,l}^* \rho} R(\pi_1(\Sigma_{l-1}), SL(2, \mathbb{C}))$; by induction, all the coefficients on the two first lines of (9) are 0. We can then exchange the roles of Σ_{l-1} and S_{v_l} to prove that the remaining coefficients are also 0, and

thus that the family $(\Re(\text{tr}_{\mu_k^l}), \Im(\text{tr}_{\mu_k^l}))_{1 \leq k \leq I_l} \cup (i_{v,l} * (f_k^v))_{v \in V_l, 1 \leq k \leq 2d(v)-6}$ has independent derivatives on the cotangent space $T_{i_l^* \rho}^* R(\pi_1(\Sigma_l), SL(2, \mathbb{C}))$.

Case 2: Σ_l is obtained by gluing Σ_{l-1} along two of its boundary components. Let us denote by $\mu_1 = \mu_{k_l}^{l-1}$ and $\mu_2 = \mu_{k'_l}^{l-1}$ the identified boundary components of Σ_{l-1} (taken with matching orientations), and by $\mu = \mu_{k_l}^l$ the resulting curve on Σ_l . The fundamental group of Σ_l is obtained as an HNN-extension of $\pi_1(\Sigma_{l-1})$; more precisely, if $\langle G | R \rangle$ is a presentation of $\pi_1(\Sigma_{l-1})$, and γ_1, γ_2 are two elements of $\pi_1(\Sigma_{l-1})$ corresponding to μ_1 and μ_2 respectively, then a presentation of $\pi_1(\Sigma_l)$ is given by $\langle G, t \mid R, t\gamma_1 t^{-1} = \gamma_2 \rangle$, and we have the following identification:

$$\begin{aligned} R(\pi_1(\Sigma_l), SL(2, \mathbb{C})) &\xrightarrow{\sim} \{(\sigma, B) \in R(\pi_1(\Sigma_{l-1}), SL(2, \mathbb{C})) \times SL(2, \mathbb{C}) ; B\sigma(\gamma_1)B^{-1} = \sigma(\gamma_2)\} \\ \varrho &\mapsto (i_{l-1,l}^*(\varrho), \varrho(t)) \end{aligned}$$

In particular, the image $i_{l-1,l}^*(R(\pi_1(\Sigma_l), SL(2, \mathbb{C})))$ is equal to the set

$$\{\sigma \in R(\pi_1(\Sigma_{l-1}), SL(2, \mathbb{C})) ; \sigma(\gamma_1) \text{ is conjugated to } \sigma(\gamma_2)\}.$$

But we know that two elements of $SL(2, \mathbb{C})$ whose traces are different from ± 2 are conjugated if and only if their traces are equal. Furthermore, $i_{l-1,l}^* \rho(\mu_1)$ and $i_{l-1,l}^* \rho(\mu_2)$ have the same trace which is not ± 2 , since they both correspond to the holonomy around the same singular edge, which is an elliptic isometry different from the identity. Consequently, if U is a small enough neighborhood of $i_{l-1,l}^*(\rho)$ in $R(\pi_1(\Sigma_{l-1}), SL(2, \mathbb{C}))$, then the set $\{\sigma \in U ; \text{tr}_{\mu_1}(\sigma) = \text{tr}_{\mu_2}(\sigma)\}$ is a neighborhood of $i_{l-1,l}^*(\rho) = i_{l-1,l}^* \rho \circ i_l^*(\rho)$ in $i_{l-1,l}^*(R(\pi_1(\Sigma_l), SL(2, \mathbb{C})))$. In particular,

$$\begin{aligned} i_{l-1,l}^*(T_{i_l^*(\rho)}^* R(\pi_1(\Sigma_l), SL(2, \mathbb{C}))) &= \{\dot{\sigma} \in T_{i_{l-1,l}^* \rho}^* R(\pi_1(\Sigma_{l-1}), SL(2, \mathbb{C})) ; d\Re \text{tr}_{\mu_1}(\dot{\sigma}) = d\Re \text{tr}_{\mu_2}(\dot{\sigma}) \\ &\quad \text{and } d\Im \text{tr}_{\mu_1}(\dot{\sigma}) = d\Im \text{tr}_{\mu_2}(\dot{\sigma})\} \end{aligned}$$

Now suppose there exists a dependence relation between the derivatives at $i_l^*(\rho)$:

$$\sum_{1 \leq k \leq I_l} (\lambda_k^1 d\Re(\text{tr}_{\mu_k^l}) + \lambda_k^2 d\Im(\text{tr}_{\mu_k^l})) + \sum_{v \in V_l} \sum_{1 \leq k \leq 2d(v)-6} \lambda_{v,k} d(i_{v,l} * (f_k^v)) = 0.$$

We can rewrite this relation as

$$\begin{aligned} i_{l-1,l} * \left(a d\Re(\text{tr}_{\mu_1}) + b d\Im(\text{tr}_{\mu_1}) + \sum_{\substack{1 \leq k \leq I_{l-1} \\ k \neq k_l, k'_l}} (\lambda_k^1 d\Re(\text{tr}_{\mu_k^{l-1}}) + \lambda_k^2 d\Im(\text{tr}_{\mu_k^{l-1}})) \right. \\ \left. + \sum_{v \in V_{l-1}} \sum_{1 \leq k \leq 2d(v)-6} \lambda_{v,k} d(i_{v,l-1} * (f_k^v)) \right) = 0 \end{aligned}$$

Pulling back this relation, we obtain that on $i_{l-1,l}^*(T_{i_l^*(\rho)}^* R(\pi_1(\Sigma_l), SL(2, \mathbb{C})))$, i.e. for all tangent vector $\dot{\sigma} \in T_{i_{l-1,l}^* \rho}^* R(\pi_1(\Sigma_{l-1}), SL(2, \mathbb{C}))$ such that $d(\Re \text{tr}_{\mu_1})(\dot{\sigma}) = d(\Re \text{tr}_{\mu_2})(\dot{\sigma})$ and $d(\Im \text{tr}_{\mu_1})(\dot{\sigma}) = d(\Im \text{tr}_{\mu_2})(\dot{\sigma})$, we have

$$\begin{aligned} a d\Re(\text{tr}_{\mu_1}) + b d\Im(\text{tr}_{\mu_1}) + \sum_{\substack{1 \leq k \leq I_{l-1} \\ k \neq k_l, k'_l}} (\lambda_k^1 d\Re(\text{tr}_{\mu_k^{l-1}}) + \lambda_k^2 d\Im(\text{tr}_{\mu_k^{l-1}})) \\ + \sum_{v \in V_{l-1}} \sum_{1 \leq k \leq 2d(v)-6} \lambda_{v,k} d(i_{v,l-1} * (f_k^v)) = 0 \end{aligned}$$

But by induction, we know that the derivatives of the family $(d\Re(\text{tr}_{\mu_k^{l-1}}), d\Im(\text{tr}_{\mu_k^{l-1}}))_{1 \leq k \leq I_{l-1}} \cup (d(i_{v,l-1} * f_k^v))_{v \in V_{l-1}, 1 \leq k \leq 2d(v)-6}$ are linearly independent on $T_{i_{l-1}^* \rho}^* R(\pi_1(\Sigma_{l-1}), SL(2, \mathbb{C}))$, so for any differential df in the above relation, we can find $\dot{\sigma} \in i_{l-1,l}^*(T_{i_l^* \rho}^* R(\pi_1(\Sigma_l), SL(2, \mathbb{C})))$ such that $df \dot{\sigma} = 1$ and $\dot{\sigma}$ is in the kernel of all the other appearing differentials. This implies that all the coefficients in the dependence relation are equal to zero, and finally the independence of the family $(\Re(\text{tr}_{\mu_k^l}), \Im(\text{tr}_{\mu_k^l}))_{1 \leq k \leq I_l} \cup (i_{v,l} * (f_k^v))_{v \in V_l, 1 \leq k \leq 2d(v)-6}$.

We have thus proved by induction that the derivatives of the functions $(\Re(\text{tr}_{\mu_k^l}), \Im(\text{tr}_{\mu_k^l}))_{1 \leq k \leq I_l} \cup (i_{v,l} * (f_k^v))_{v \in V_l, 1 \leq k \leq 2d(v)-6}$ are linearly independent on $T_{i_l^* \rho}^* R(\pi_1(\Sigma_l), SL(2, \mathbb{C}))$. For $l = |E|$, we obtain that the functions of the family $(\Re(\text{tr}_{\mu_e}), \Im(\text{tr}_{\mu_e}))_{e \in E} \cup (f_k^v)_{v \in V, 1 \leq k \leq 2d(v)-6}$ have linearly independent derivatives on $T_\rho^* R(\pi_1(\Sigma_\epsilon^c), SL(2, \mathbb{C}))$, provided that ρ is a smooth point of $R(\pi_1(\Sigma_\epsilon^c), SL(2, \mathbb{C}))$. If this is true, $[\rho]$ is a smooth point of $X(\pi_1(\Sigma_\epsilon^c), SL(2, \mathbb{C}))$, since by Corollary 10 it is irreducible. Then the functions of the family $(\Re(\text{tr}_{\mu_e}), \Im(\text{tr}_{\mu_e}))_{e \in E} \cup (f_k^v)_{v \in V, 1 \leq k \leq 2d(v)-6}$ have linearly independent derivatives on $T_{[\rho]}^* X(\pi_1(\Sigma_\epsilon^c), SL(2, \mathbb{C}))$, i.e. the map F^c is locally an immersion, and its level sets are local submanifolds.

So we will now prove that the representation space $R(\pi_1(\Sigma_\epsilon^c), SL(2, \mathbb{C}))$ is smooth at ρ . We have seen that $\Sigma_\epsilon^c = \Sigma_{|E|}$ is obtained by gluing $\Sigma_{|E|-1}$ along its two boundary components, so that $\pi_1(\Sigma_\epsilon^c)$ is a HNN-extension of $\pi_1(\Sigma_{|E|-1})$, and $R(\pi_1(\Sigma_\epsilon^c), SL(2, \mathbb{C}))$ can be identified to the subspace of $R(\pi_1(\Sigma_{|E|-1}), SL(2, \mathbb{C})) \times SL(2, \mathbb{C}) \simeq SL(2, \mathbb{C})^{n+1}$ consisting of pairs (σ, B) satisfying $B\sigma(\gamma_1)B^{-1}\sigma(\gamma_2)^{-1} = Id$. More precisely, if we introduce the following function

$$\begin{aligned} f : R(\pi_1(\Sigma_{|E|-1}), SL(2, \mathbb{C})) \times SL(2, \mathbb{C}) &\rightarrow SL(2, \mathbb{C}) \\ (\sigma, B) &\mapsto B\sigma(\gamma_1)B^{-1}\sigma(\gamma_2)^{-1} \end{aligned}$$

then $R(\pi_1(\Sigma_\epsilon^c), SL(2, \mathbb{C}))$ is the preimage of the identity. It is thus sufficient to show that f is a submersion at $(i_{|E|-1}^* \rho, \rho(t))$, where t , as above, is the new generator of $\pi_1(\Sigma_\epsilon)$. We compute that

$$df_{(i_{|E|-1}^* \rho, \rho(t))}(\dot{\sigma}, \dot{B}) = \dot{B} + Ad(\rho(t))\dot{\sigma}(\gamma_1) - Ad(i_{|E|-1}^* \rho(\gamma_2))\dot{B} - \dot{\sigma}(\gamma_2),$$

where \dot{B} belongs to the set $\mathfrak{sl}(2, \mathbb{C})$ of traceless matrices and $\dot{\sigma} : \pi_1(\Sigma_{|E|-1}) \rightarrow \mathfrak{sl}(2, \mathbb{C})$ satisfies the cocycle condition (8). Let $v \in \mathfrak{sl}(2, \mathbb{C})$; we want to prove that there exists a pair $(\dot{\sigma}, \dot{B})$ such that $df_{(i_{|E|-1}^* \rho, \rho(t))}(\dot{\sigma}, \dot{B}) = v$. This can be rewritten as

$$\dot{B} - Ad(i_{|E|-1}^* \rho(\gamma_2))\dot{B} = v + \dot{\sigma}(\gamma_2) - Ad(\rho(t))\dot{\sigma}(\gamma_1).$$

Now for any $M \in SL(2, \mathbb{C})$, $M \neq Id$, the image of the endomorphism $m \mapsto m - Ad(M)m$ of $\mathfrak{sl}(2, \mathbb{C})$ is the set $\{w \in \mathfrak{sl}(2, \mathbb{C}); \text{tr}(wM) = 0\}$. We can apply this elementary result to $i_{|E|-1}^* \rho(\gamma_2)$, which is not the identity because it corresponds to the holonomy around a singular edge. So there exists a solution \dot{B} of the above equation if and only if $\text{tr}(w i_{|E|-1}^* \rho(\gamma_2)) = 0$, where $w = v + \dot{\sigma}(\gamma_2) - Ad(\rho(t))\dot{\sigma}(\gamma_1)$ is the right hand side term; this gives a condition on $\dot{\sigma}$. We have just proven the independence of the functions $\text{tr}_{\gamma_1} = \text{tr}_{\mu_1}$ and $\text{tr}_{\gamma_2} = \text{tr}_{\mu_2}$ near $i_{|E|-1}^* \rho$ on $R(\pi_1(\Sigma_{|E|-1}), SL(2, \mathbb{C}))$, so there exists $\dot{\sigma} \in T_{i_{|E|-1}^* \rho}^* R(\pi_1(\Sigma_{|E|-1}), SL(2, \mathbb{C}))$ such that $d\text{tr}_{\gamma_2} \dot{\sigma} = -\text{tr}(v i_{|E|-1}^* \rho(\gamma_2))$ and $d\text{tr}_{\gamma_1} \dot{\sigma} = 0$. But $d\text{tr}_{\gamma_2} \dot{\sigma} = \text{tr}(\dot{\sigma}(\gamma_2) i_{|E|-1}^* \rho(\gamma_2))$ and similarly $d\text{tr}_{\gamma_1} \dot{\sigma} = \text{tr}(\dot{\sigma}(\gamma_1) i_{|E|-1}^* \rho(\gamma_1))$. Then

$$\begin{aligned} \text{tr}(w i_{|E|-1}^* \rho(\gamma_2)) &= \text{tr}(v i_{|E|-1}^* \rho(\gamma_2)) + \text{tr}(\dot{\sigma}(\gamma_2) i_{|E|-1}^* \rho(\gamma_2)) - \text{tr}(Ad(\rho(t))\dot{\sigma}(\gamma_1) i_{|E|-1}^* \rho(\gamma_2)) \\ &= \text{tr}(v i_{|E|-1}^* \rho(\gamma_2)) + \text{tr}(\dot{\sigma}(\gamma_2) i_{|E|-1}^* \rho(\gamma_2)) - \text{tr}(\dot{\sigma}(\gamma_1) i_{|E|-1}^* \rho(\gamma_1)) \\ &= \text{tr}(v i_{|E|-1}^* \rho(\gamma_2)) + d\text{tr}_{\gamma_2} \dot{\sigma} - d\text{tr}_{\gamma_1} \dot{\sigma} = 0 \end{aligned}$$

Hence there exists $\dot{B} \in \mathfrak{sl}(2, \mathbb{C})$ such that $\dot{B} - Ad(i_{|E|-1}^* \rho(\gamma_2))\dot{B} = w$, and so $df_{(i_{|E|-1}^* \rho, \rho(t))}(\dot{\sigma}, \dot{B}) = v$. Consequently f is a submersion near $(i_{|E|-1}^* \rho, \rho(t))$, and ρ is a smooth point of $R(\pi_1(\Sigma_\epsilon^c), SL(2, \mathbb{C}))$.

This computation also show that this latter space has the same dimension as $R(\pi_1(\Sigma_{|E|-1}), SL(2, \mathbb{C}))$. Let us denote by g the genus of Σ_ϵ^c ; it is equal to $|E| - |V| + 1$. Then $\Sigma_{|E|-1}$ is a surface of genus $g - 1$ with two holes, so its fundamental group is a free group on $2(g - 1) + 1$ generators, and the real dimension of its representation space is $6(2g - 1) = 12g - 6$. Finally the real dimension of $X(\pi_1(\Sigma_\epsilon), SL(2, \mathbb{C}))$ is $12g - 12$, which is also equal to $12|E| - 12|V| = 2(2|E| + \sum_{v \in V} (2d(v) - 6))$. This ends the proof of Theorem 12. \square

If the singular locus Σ has a component that is a circle, then Σ_ϵ has a torus component. The corresponding case has been treated in [5] (Theorems 4.4 and 4.5) and [15] (section 6.6.1); we will just quote the result. We keep the notations of Theorem 12: ρ is the holonomy representation of a closed, hyperbolic 3-cone-manifold with cone angles smaller than 2π , Σ^c is a circle component of the singular locus, Σ_ϵ^c is a torus, boundary of a tubular neighborhood of Σ^c . The induced representation on $\pi_1(\Sigma_\epsilon^c)$ is also denoted by ρ , and μ is a simple closed curve on Σ_ϵ^c going around Σ^c , i.e. it is a meridian curve of the torus.

Theorem 13 (Torus case, [5], [15]).

With the above notations, consider the function

$$\begin{aligned} F^c : X(\pi_1(\Sigma_\epsilon^c), SL(2, \mathbb{C})) &\rightarrow \mathbb{R}^2 \\ [\varrho] &\mapsto \left(\Re \text{tr}_\mu(\varrho), \Im \text{tr}_\mu(\varrho) \right) \end{aligned}$$

Then in a neighborhood of $[\rho]$, which is a smooth point of $X(\pi_1(\Sigma_\epsilon^c), SL(2, \mathbb{C}))$, the level sets of F^c are local half-dimensional submanifolds.

4 Local deformations

As usual, let \bar{M} be a closed, orientable, hyperbolic 3-cone-manifold with cone angles smaller than 2π . Our goal in this section is to find local coordinates near $[\rho]$ on the space $X(\pi_1(M), SL(2, \mathbb{C}))$; this will be achieved by lifting the functions defined in the previous section. But first we need to know that $X(\pi_1(M), SL(2, \mathbb{C}))$ is smooth near $[\rho]$. This is indeed the case according to the following theorem, which can be found in M. Kapovich's book [7]; see also [3], [13], and [15] section 6.7.1.

Theorem 14 (Smoothness of the Holonomy Representation, [7]).

Let ρ be the holonomy representation of a closed, oriented, connected hyperbolic 3-cone-manifold \bar{M} , with cone angles smaller than 2π . Then $X(\pi_1(M), SL(2, \mathbb{C}))$ is smooth at $[\rho]$, and its real dimension is $2\tau - 3\chi(\Sigma_\epsilon)$, where τ is the number of torus components in Σ_ϵ .

Let us denote by $(\Sigma^c)_{c \in \mathcal{C}}$ the connected components of Σ , the singular locus of \bar{M} , so $\Sigma = \coprod_{c \in \mathcal{C}} \Sigma^c$. The surface Σ_ϵ , which is the boundary of a tubular neighborhood of the singular locus, also decomposes as a union $\Sigma_\epsilon = \coprod_{c \in \mathcal{C}} \Sigma_\epsilon^c$. For each component of Σ_ϵ , the inclusion map $i_c : \Sigma_\epsilon^c \hookrightarrow M$ induces a map $i_c^* : X(\pi_1(M), SL(2, \mathbb{C})) \rightarrow X(\pi_1(\Sigma_\epsilon^c), SL(2, \mathbb{C}))$. We can apply Theorem 12 or Theorem 13 to each component of Σ_ϵ : this gives a family of local functions $F^c : X(\pi_1(\Sigma_\epsilon^c), SL(2, \mathbb{C})) \rightarrow \mathbb{R}^{n_c}$, where $n_c = 2$ if Σ_ϵ^c is a torus, or $n_c = 6g_c - 6$ if Σ_ϵ^c has genus $g_c \geq 2$. These functions can be lifted to local functions $i_{c*} F_c$ on $X(\pi_1(M), SL(2, \mathbb{C}))$ in a neighborhood of $[\rho]$. Note that the real dimension of $X(\pi_1(M), SL(2, \mathbb{C}))$ is exactly equal to $\sum_{c \in \mathcal{C}} n_c$.

Theorem 15. *Let ρ be the holonomy representation of a closed, oriented, connected hyperbolic 3-cone-manifold \bar{M} , with singular locus $\Sigma = \coprod_{c \in \mathcal{C}} \Sigma^c$ and cone angles smaller than 2π . Then the local*

function

$$\begin{aligned} F = (i_{c*} F^c)_{c \in \mathcal{C}} : X(\pi_1(M), SL(2, \mathbb{C})) &\rightarrow \bigoplus_{c \in \mathcal{C}} \mathbb{R}^{n_c} \\ [\varrho] &\mapsto (F^c(i_c^*[\varrho]))_{c \in \mathcal{C}} \end{aligned}$$

is a coordinate chart of $X(\pi_1(M), SL(2, \mathbb{C}))$ in a neighborhood of $[\rho]$.

Proof. Let us consider the restriction map:

$$\begin{aligned} r : X(\pi_1(M), SL(2, \mathbb{C})) &\rightarrow \bigtimes_{c \in \mathcal{C}} X(\pi_1(\Sigma_\epsilon^c), SL(2, \mathbb{C})) \\ [\varrho] &\mapsto (i_c^*[\varrho])_{c \in \mathcal{C}} \end{aligned}$$

We have seen in section 2 that the Zariski tangent space at $[\rho]$ (resp. $i_c^*[\rho]$) of $X(\pi_1(M), SL(2, \mathbb{C}))$ (resp. $X(\pi_1(\Sigma_\epsilon^c), SL(2, \mathbb{C}))$) is identified with the first cohomology group $H^1(M; \mathcal{E})$ (resp. $H^1(\Sigma_\epsilon^c; \mathcal{E})$); and since $[\rho]$ (resp. $i_c^*[\rho]$) is a smooth point, this is also the usual tangent space. The tangent map Tr at $[\rho]$ is thus identified with the natural map

$$H^1(M; \mathcal{E}) \rightarrow \bigoplus_{c \in \mathcal{C}} H^1(\Sigma_\epsilon^c; \mathcal{E}) = H^1(\Sigma_\epsilon; \mathcal{E}) = H^1(U_\epsilon; \mathcal{E}).$$

According to Proposition 7, this is an injective map, with half-dimensional image. Hence r is an immersion near $[\rho]$, and we just have to prove that its image is locally transverse to the level sets of the function F . Let us denote by L the level set of the function F passing through $[\rho]$; its tangent space $T_{[\rho]}L$ is identified to a subspace of $H^1(U_\epsilon; \mathcal{E})$. It is sufficient to show that the image of Tr is in direct sum with $T_{[\rho]}L$; because of dimensions, it is actually enough to prove that $T_{[\rho]}L$ and the image of Tr , i.e. the image of $H^1(M; \mathcal{E})$ in $H^1(U_\epsilon; \mathcal{E})$, have trivial intersection.

We will begin by exhibiting a basis of $T_{[\rho]}L$ whose elements have “nice” representatives as closed \mathcal{E} -valued 1-forms over U_ϵ , and we will compute the corresponding infinitesimal deformations of the metric tensor. We will then be able to use Theorem 6 to conclude the proof.

For each vertex $v \in V$ of valence $d(v)$, we have seen that the regular part L_v of the link of v is a $d(v)$ -punctured sphere, which embeds as S_v in Σ_ϵ . Recall that v admits a neighborhood \bar{U}_v that is a cone over \bar{L}_v , in which the metric can be expressed in spherical coordinates as (3):

$$g = dr^2 + \sinh(r)^2 g_{\bar{L}_v}, \quad r \in [0, \epsilon)$$

The regular part U_v of \bar{U}_v admits S_v as a deformation retract, hence $H^1(U_v; \mathcal{E}) = H^1(S_v; \mathcal{E})$.

According to Proposition 11, the family $(\Re \text{tr } \mu_k^v, \Im \text{tr } \mu_k^v)_{1 \leq k \leq d(v)} \cup (f_k^v, g_k^v)_{1 \leq k \leq 2d(v)-6}$ is a local coordinate system near $[i_v^* \rho]$ on $X(\pi_1(S_v), SL(2, \mathbb{C}))$. For each k , $1 \leq k \leq 2d(v) - 6$, we can consider the vector $\frac{\partial}{\partial g_k^v} \in T_{[i_v^* \rho]} X(\pi_1(S_v), SL(2, \mathbb{C})) \simeq H^1(S_v; \mathcal{E})$. This first-order perturbation preserves the trace of the boundary elements, hence their conjugacy classes; consequently, as an element of $H^1(S_v; \mathcal{E})$ it has a compactly supported representative $\omega_k^v \in Z^1(S_v; \mathcal{E})$. Since \mathcal{E} is flat, ω_k^v can be extended straightforwardly to $Z^1(U_v; \mathcal{E})$ by making its coefficients independent of the r variable, and since it is supported away from the edges of U_v , it can be prolonged by zero in the remainder of U_ϵ , yielding an element of $Z^1(U_\epsilon; \mathcal{E})$; it is clear from its definition that its class $[\omega_k^v]$ belongs to $T_{[\rho]}L$.

We can say more about ω_k^v . We know that $[i_v^* \rho]$ actually belongs to $X(\pi_1(S_v), SU(2, \mathbb{C}))$: it is the holonomy representation of the spherical cone-surface structure on \bar{L}_v . According to Proposition 11, the vector $\frac{\partial}{\partial g_k^v}$ lies in $T_{[i_v^* \rho]} X(\pi_1(S_v), SU(2, \mathbb{C}))$, seen as a subset of $T_{[i_v^* \rho]} X(\pi_1(S_v), SL(2, \mathbb{C}))$. This means that ω_k^v corresponds to a (compactly supported) deformation of the spherical structure on L_v . In particular, it induces a compactly supported, curvature-preserving infinitesimal deformation $h_{L_v}^k$ of $g_{\bar{L}_v}$. Finally, the induced infinitesimal deformation h_k^v of the metric on U_ϵ expresses as $h_k^v = \sinh(r)^2 h_{L_v}^k$.

in U_v and is 0 elsewhere. It is supported away from the singular edges, and we can check that its pointwise norm is bounded, whereas that of ∇h_k^v is not but is equal to $c \cdot \sinh(r)^{-1}$ as r goes to 0; hence both h_k^v and ∇h_k^v are in L^2 .

Now for each singular edge e (which has possibly no ends, i.e. is a circular component of Σ), we can choose a point p on e which has a neighborhood that do not intersect the support of any of the above closed forms. Near p the metric can be expressed in local cylindrical coordinates as (4):

$$g = d\rho^2 + \sinh(\rho)^2 d\theta^2 + \cosh(\rho)^2 dz^2, \quad \rho \in [0, \epsilon), \quad \theta \in \mathbb{R}/\alpha\mathbb{Z}, \quad z \in (-\epsilon, \epsilon)$$

The vector fields $\frac{\partial}{\partial z}$ and $\frac{\partial}{\partial \theta}$ are local Killing vector fields and thus correspond to local parallel sections $\sigma_{\partial/\partial z}$ and $\sigma_{\partial/\partial \theta}$ of \mathcal{E} . Let $\phi : (-\epsilon, \epsilon) \rightarrow [0, 1]$ be a smooth function such that $\phi(z) = 0$ if $z \leq -\epsilon/2$ and $\phi(z) = 1$ if $z \geq \epsilon/2$. The following \mathcal{E} -valued 1-forms on U_e are then closed:

$$\omega_z^e = d\phi \wedge \sigma_{\partial/\partial z}$$

$$\omega_\theta^e = d\phi \wedge \sigma_{\partial/\partial \theta}$$

Being supported away from the vertices, they do not deform the holonomy representation of the link of any vertex, and in particular their classes belong to $T_{[\rho]}L$. More precisely, ω_z^e corresponds to a change of the length of e while ω_θ^e corresponds to a change of the “gluing parameter” along e , cf. section 3 of [9]. The associated infinitesimal deformations of the metric tensors are quite easy to compute; we find that $h_z^e = \phi'(z) \cosh(\rho)^2 dz^2$ and $h_\theta^e = \phi'(z) \sinh(\rho)^2 d\theta dz$, and that both are L^2 , with L^2 covariant derivatives.

Because of the dimension of L , if the family $(\omega_k^v)_{v \in V, 1 \leq k \leq 2d(v)-6} \cup (\omega_z^e, \omega_\theta^e)_{e \in E}$ is linearly independent in $H^1(U_\epsilon; \mathcal{E})$ then it forms a basis of $T_{[\rho]}L$. So, suppose there exists a linear combination of these forms which is a coboundary:

$$\sum_{v,k} \lambda_k^v \omega_k^v + \sum_e (\lambda_z^e \omega_z^e + \lambda_\theta^e \omega_\theta^e) = ds$$

for some section s of \mathcal{E} . Over U_v , we get $\sum_k \lambda_k^v \omega_k^v = ds$ (the other forms being supported away from U_v), which implies $\lambda_k^v = 0$ for $1 \leq 2d(v) - 6$ because the forms ω_k^v are independent in $H^1(U_v; \mathcal{E}) \simeq T_{[i_v^* \rho]} X(\pi_1(S_v), SL(2, \mathbb{C}))$. So $ds = 0$ over S_v , but since $i_v^* \rho$ is irreducible there exists no non-zero constant section of \mathcal{E} over S_v . Therefore $s = 0$ over U_v , and this being true for any v , finally $s = 0$ away from the support of the family $(\omega_z^e, \omega_\theta^e)_{e \in E}$. Now on a neighborhood of a singular edge e , we have $ds = \lambda_z^e \omega_z^e + \lambda_\theta^e \omega_\theta^e = \lambda_z^e d\phi \wedge \sigma_{\partial/\partial z} + \lambda_\theta^e d\phi \wedge \sigma_{\partial/\partial \theta}$. Integrating ds along a path parallel to e , we obtain $0 = \lambda_z^e \sigma_{\partial/\partial z} + \lambda_\theta^e \sigma_{\partial/\partial \theta}$, thus $\lambda_z^e = \lambda_\theta^e = 0$. So the family $(\omega_k^v)_{v \in V, 1 \leq k \leq 2d(v)-6} \cup (\omega_z^e, \omega_\theta^e)_{e \in E}$ is indeed linearly independent in $H^1(U_\epsilon; \mathcal{E})$, hence a basis of $T_{[\rho]}L$.

We can now finish the proof. Let $[\omega] \in H^1(M; \mathcal{E})$ be such that $Tr([\omega])$ belongs to $T_{[\rho]}L$. So $Tr([\omega])$ has a representative ω' which is a linear combination of the above forms. But since Tr is just the restriction to U_ϵ , this means that $[\omega]$ also has a representative ω which is over U_ϵ a linear combination of the above forms. The corresponding infinitesimal deformation h of the metric tensor is then in L^2 , together with its covariant derivative ∇h . So we can apply the rigidity result of Theorem 6, which states that h is trivial; thus ω is a coboundary and $[\omega] = 0$. Consequently $T_{[\rho]}L$ and the image of Tr are in direct sum, which implies that the image of r and the level sets of the function F are locally transverse. \square

Now that we have established the existence of this local coordinate system, we are in a position to prove the two main results of this article:

Theorem 16. *Let \bar{M} be a closed, orientable, hyperbolic 3-cone-manifold with singular locus Σ and cone angles smaller than 2π . Then the tuple $(\alpha_e)_{e \in E}$ of cone angles gives a local parameterization of the space of closed hyperbolic 3-cone manifolds with singular locus Σ in a neighborhood of \bar{M} .*

Proof. As usual, let ρ be the holonomy representation of M . We know that ρ is irreducible (Corollary 10), so according to Theorem 4 (from [4]), the space of (equivalence classes of) hyperbolic structures on the regular part M of \bar{M} in a neighborhood of the given cone-manifold structure is homeomorphic to a neighborhood of $[\rho]$ in $X(\pi_1(M), SL(2, \mathbb{C}))$. By the preceding corollary, it admits as local coordinates the functions $(\Re \text{tr}_{\mu_e}, \Im \text{tr}_{\mu_e})_{e \in E}$ and $(f_k^v)_{v \in V, 1 \leq k \leq 2d(v)-6}$. But most of these neighboring hyperbolic structures are not cone-manifolds ones. Locally, the set of cone-manifold structures with singular locus Σ on M corresponds to the set of representations such that the holonomy of any vertex's link is contained in a maximal compact subgroup of $SL(2, \mathbb{C})$, i.e. is contained in $SU(2, \mathbb{C})$ up to conjugacy, and such that the holonomies of the edges are elliptic isometries. As we have seen in Proposition 11, this is equivalent to the conditions $\Im \text{tr}_{\mu_e} = 0$, $e \in E$ and $f_k^v = 0$, $v \in V, 1 \leq k \leq 2d(v) - 6$. Therefore the space of cone-manifold structures with singular locus Σ is locally parameterized by the tuple of the (real) traces of the singular edges' holonomy. Now if α is the cone angle of an edge, then the corresponding trace is equal to $\pm 2 \cos(\alpha/2)$; thus the cone angle tuple yields another valid parameterization. \square

Remark that if Σ has some vertices of valence greater than 3, then M most certainly admits other nearby cone-manifold structures than those mentioned above, but their singular loci will be different from Σ . As discussed in the introduction, this is because some vertices may “split” into several lower valence vertices, see figure 1.

We need some more definitions before turning our attention to the proof of Stoker problem. We will say that two convex polyhedra \mathcal{P}_1 and \mathcal{P}_2 have the same combinatorial type if there exists an oriented homeomorphism $\mathcal{P}_1 \rightarrow \mathcal{P}_2$ which sends faces to faces, edges to edges and vertices to vertices. Equivalently, \mathcal{P}_1 and \mathcal{P}_2 have the same combinatorial type if there exists a bijection f between the sets of vertices of \mathcal{P}_1 and \mathcal{P}_2 , such that two vertices bound an edge in \mathcal{P}_1 if and only if their images bound an edge in \mathcal{P}_2 , and a family of vertices of \mathcal{P}_1 is the set of vertices of a face if and only if its image is the set of vertices of a face of \mathcal{P}_2 . Such a map will be called a marking of \mathcal{P}_2 by \mathcal{P}_1 .

Now let us fix a convex polyhedron \mathcal{P} . A marked polyhedron having the combinatorial type of \mathcal{P} is a couple (\mathcal{Q}, f) where f is a marking of \mathcal{Q} by \mathcal{P} . We define $\overline{Pol}(\mathcal{P})$ as the set of convex marked polyhedra having the combinatorial type of \mathcal{P} . The direct isometry group of the ambient space acts freely on $\overline{Pol}(\mathcal{P})$; the quotient is denoted by $Pol(\mathcal{P})$. The reason for introducing marked polyhedra is that the space of congruence class of convex polyhedra with the combinatorial type of \mathcal{P} is generally an orbifold, whereas its ramified cover $Pol(\mathcal{P})$ is a smooth manifold.

Lemma 17. *Let \mathcal{P} be a closed (strictly) convex polyhedron in \mathbb{H}^3 , having $|E|$ edges. Then $Pol(\mathcal{P})$ is a manifold of dimension $|E|$.*

Proof. We will begin by proving this result for convex polyhedra in Euclidean space. Let us denote by V (resp. E and F) the set of vertices (resp. edges and faces) of \mathcal{P} . For each face $f \in F$, denote by v_f^1, v_f^2 and v_f^3 three of its vertices, consecutive along its oriented boundary. An element of $\overline{Pol}(\mathcal{P})$, that is, a convex Euclidean marked polyhedron having the combinatorial type of \mathcal{P} , is then a collection $(x_v)_{v \in V} \in \mathbb{R}^{3|V|}$ of points satisfying the following conditions :

1. for each face $f \in F$, for each vertex $v \in f \setminus \{v_f^1, v_f^2, v_f^3\}$, $\det(x_{v_f^2} - x_{v_f^1}, x_{v_f^3} - x_{v_f^1}, x_v - x_{v_f^1}) = 0$ (planarity of the faces),
2. for each face $f \in F$, for each vertex $v \in V$, $v \notin f$, $\det(x_{v_f^2} - x_{v_f^1}, x_{v_f^3} - x_{v_f^1}, x_v - x_{v_f^1}) > 0$ (strict convexity).

One can check that under condition 2, the functions involved in condition 1 are independent (i.e. have linearly independent derivatives), so both conditions define a (possibly empty) submanifold of $\mathbb{R}^{3|V|}$.

Let us denote by d_f the number of vertices in a face f . We can then compute the dimension of $\overline{Pol}(\mathcal{P})$: it is $3|V| - \sum_{f \in F} (d_f - 3) = 3|V| + 3|F| - \sum_{f \in F} d_f$. But d_f is also the number of edges in a given face, and since each edge belongs to exactly two faces, $\sum_{f \in F} d_f = 2|E|$, so the dimension is equal to $3|V| + 3|F| - 2|E|$. The Euler characteristic of a convex polyhedron being 2, (i.e. $|V| + |F| - |E| = 2$), we deduce that $\dim \overline{Pol}(\mathcal{P}) = |E| + 6$.

Now let us go back to \mathbb{H}^3 . We will use the projective (or Klein) model to embed \mathbb{H}^3 as the open unit ball of \mathbb{R}^3 ; this embedding is not conformal, but maps geodesic lines and planes of \mathbb{H}^3 to (portions of) straight lines and planes. In particular, it identifies convex hyperbolic polyhedra with convex Euclidean polyhedra contained in the open unit ball, and of course it preserves the combinatorial type. This shows that the above dimension computation is also valid for convex hyperbolic polyhedra. Finally, since the action of $\text{Isom}^+(\mathbb{H}^3)$ on $\overline{Pol}(\mathcal{P})$ is free and proper, the quotient $Pol(\mathcal{P})$ is a manifold of dimension $\dim \overline{Pol}(\mathcal{P}) - \dim \text{Isom}^+(\mathbb{H}^3) = |E|$. \square

Theorem 18. *Let \mathcal{P} be a closed (strictly) convex polyhedron in \mathbb{H}^3 , having $|E|$ edges. Then the tuple $(\alpha_1, \dots, \alpha_N)$ of dihedral angles gives a local parameterization of $Pol(\mathcal{P})$.*

Proof. Let us denote by $\bar{M} = D(\mathcal{P})$ the double of \mathcal{P} ; this is a closed hyperbolic 3-cone-manifold, whose cone angles are smaller than 2π because \mathcal{P} is convex, and its singular locus Σ corresponds to the graph formed by the edges and the vertices of \mathcal{P} . Let $C(M, \Sigma)$ be the space of cone-manifold structures with singular locus Σ on the regular part M of $D(\mathcal{P})$. The double construction yields an injective map

$$Pol(\mathcal{P}) \rightarrow C(M, \Sigma) \quad (10)$$

Now according to Theorem 16 the space $C(M, \Sigma)$ is near $D(\mathcal{P})$ a manifold whose dimension is equal to the number of its singular edges. Hence (10) is near \mathcal{P} an injective map between two manifolds of the same dimension, and thus a local diffeomorphism. The local parameterization of $C(M, \Sigma)$ by the cone angles therefore yields a parameterization of $Pol(\mathcal{P})$ by the dihedral angles. \square

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